Multi-legs, Superfluids & Semiclassics

Riccardo Rattazzi, EPFL

with
Gil Badel, Gabriel Cuomo, Alexander Monin, arXiv:1909.01269, arXiv:1911.08505

Extremal Correlators and Random Matrix Theory
Alba Grassi, Zohar Komargodski, Luigi Tizzano. Aug 27, 2019. 49 pp.
e-Print: **arXiv:1908.10306**

The large charge limit of scalar field theories and the Wilson-Fisher fixed point at ϵ=0 ϵ = 0
G. Arias-Tamargo, D. Rodriguez-Gomez, J.G. Russo. Aug 29, 2019. 14 pp.
e-Print: **arXiv:1908.11347**

Accessing Large Global Charge via the ϵ ϵ -Expansion
Masataka Watanabe. Sep 3, 2019. 15 pp.
e-Print: **arXiv:1909.01337**

[Weak vs Strong] & [Classical vs Quantum]

▲ Weak coupling: loop expansion around leading trajectory $\gamma_{c\ell}$

$$e^{-W} = e^{-[S_0+S_1+S_2+\dots]}$$

can further distinguish semi-classical and quantum observables

$$\langle O \rangle = O(\gamma_{c\ell}) + \delta_q O$$

$\delta_q O \ll O(\gamma_{c\ell})$ semi-class — Ex: every day life

$\delta_q O \gtrsim O(\gamma_{c\ell})$ quantum — Ex: $\langle \phi\phi\phi\phi \rangle$ around $\phi(\gamma_{c\ell}) = 0$

▲ Strong coupling: PI cannot be described by leading trajectory

Common practice: few legs in weakly coupled QFT

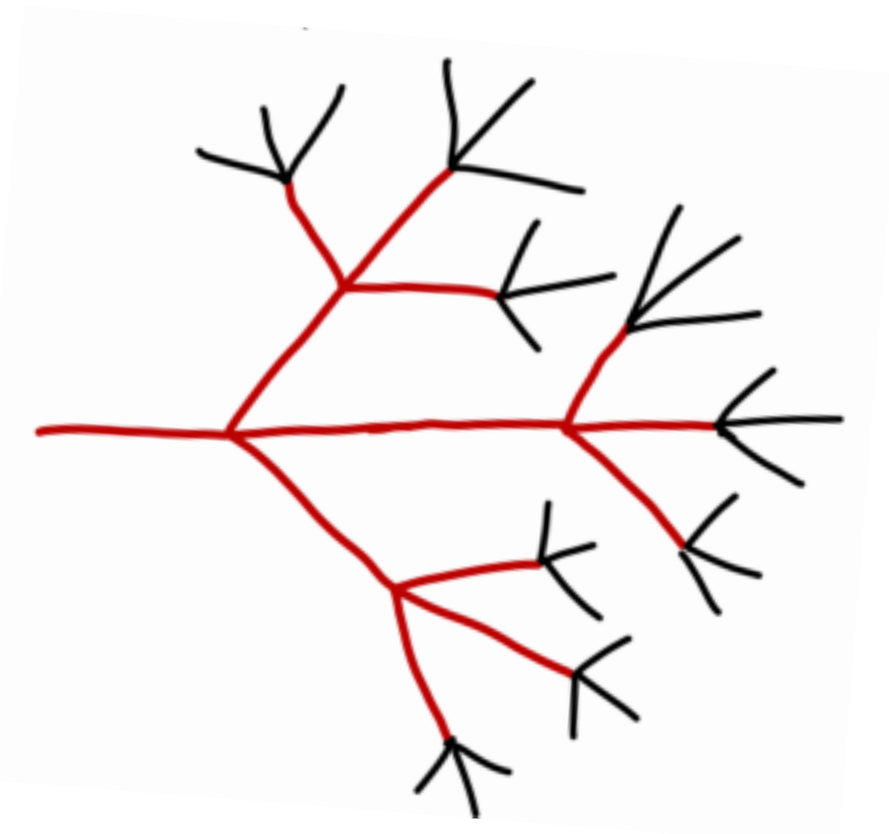
= small fluctuations around trivial trajectory

However when the number of legs grows expansion breaks down

How do we describe physics in this regime?

n-legs in ϕ^4

see old review by Rubakov, arXiv:9511236, 1995



$$A_{1\to n} \propto n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

$$\sigma_{1\to n} \propto n! \left(\frac{\lambda}{8}\right)^{n-1} \left(\frac{\epsilon}{m}\right)^{\frac{3n-1}{2}} \qquad \epsilon = \frac{E - nm}{n}$$

$$A_{loop} = A_{tree}\left(1 + B\lambda n^2 + C\lambda^2 n^4 + \dots\right)$$

a mess ?!

Indeed .. but not completely

all large effects can be proven to resum into

$$\sigma_{1\to n} \sim e^{nF(\lambda n,\epsilon)}$$

Libanov, Rubakov, Son, Troitsky 1994

Exponential form suggest existence of non-trivial semiclassical trajectory describing the process

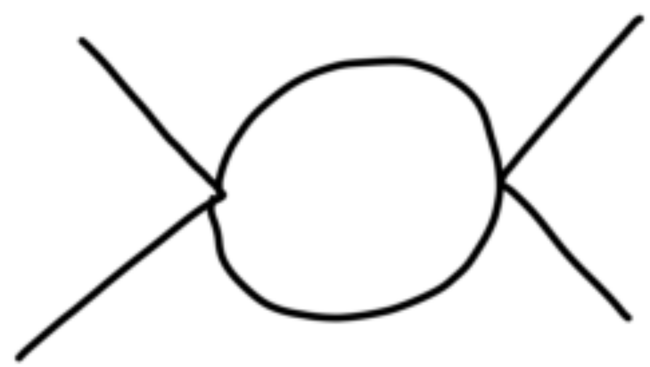
Something indeed proven by Son, back in the 90's...still a reasonable mess to work out quantitatively

see Khoze, Reiness, 1810.01722

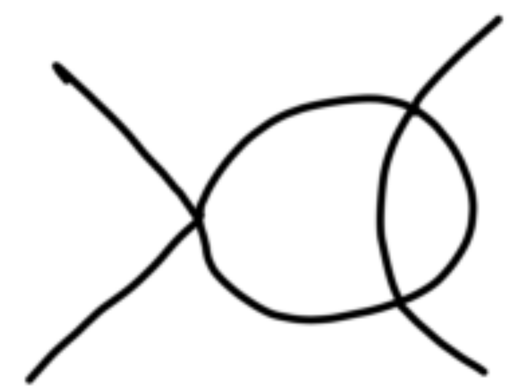
Charged ϕ^4 $\mathcal{L} = \partial_\mu \overline{\phi}\, \partial^\mu \phi + \frac{\lambda}{4}(\overline{\phi}\phi)^2$

Few Legs

λ



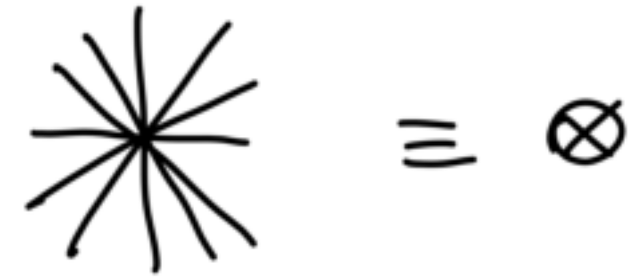
$\frac{\lambda^2}{16\pi^2}$



$\frac{\lambda^3}{(16\pi^2)^2}$

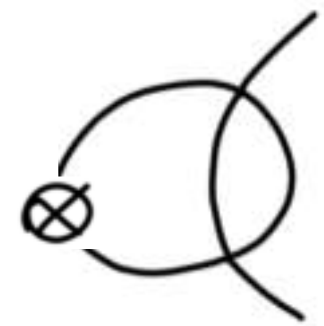
Many Legs

Ex. anomalous dimension of ϕ^n $\quad n \gg 1$

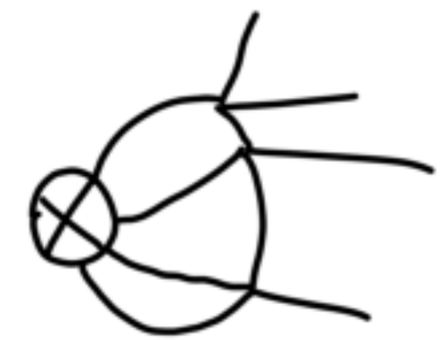


$\lambda n(n-1)$

$\lambda^2 n(n-1)(n-2)$



$\lambda^2 n(n-1)$



$\lambda^3 n(n-1)(n-2)(n-3)$

leading $= n\left[\lambda n + (\lambda n)^2 + (\lambda n)^3 + \ldots\right]$

perturbation theory breaks down at $\dfrac{\lambda n}{16\pi^2} \gtrsim 1$

$n(n-1)\,\lambda\,L$

$n(n-1)(n-2)\,\lambda^2\,(L^2+L)$

$n(n-1)(n-2)(n-3)\,\lambda^3\,(L^3+L^2+L)$

$\cdots$

$\frac{1}{2}n(n-1)(n-2)(n-3)\,\lambda^2\,L^2$

$\cdots$

$\cdots$

$\cdots$

$$\underbrace{\lambda^\ell n^{2\ell},\,\ldots,\,\lambda^\ell n^{\ell+2},}_{\text{exponentiate}}\ \lambda^\ell n^{\ell+1},\,\ldots,\,\lambda^\ell n$$

$$Z_{\phi^n} = e^{nL\left(\sum_{\kappa=0}\lambda^\kappa P_\kappa(\lambda n)+O(L)\right)}$$

$$\gamma_n = \frac{d\log Z_{\phi^n}}{dL}$$

series can be organized as a double expansion

$$\frac{\gamma_n}{n} = \frac{1}{n}\frac{d\log Z_n}{dL} = P_0(\lambda n) + \lambda\, P_1(\lambda n) + \lambda^2 P_2(\lambda n) + \ldots$$

similar to RG $\quad F_0(\lambda\, Log) + \lambda F_1(\lambda\, Log) + \ldots$

or to 't Hooft large-N expansion

$$\frac{\gamma_n}{n} = P_0(\lambda n) + \frac{1}{n}\bar{P}_1(\lambda n) + \frac{1}{n^2}\bar{P}_2(\lambda n) + \ldots$$

- Can one derive exponentiation and structure of the series without detailed diagrammatics?

- Can one resum the λn series?

Common answer:

Semiclassical expansion around non-trivial trajectory

$$\int \left[\partial\bar{\phi}\partial\phi + \frac{\lambda}{4}\left(\bar{\phi}\phi\right)^2\right] \quad \overset{\phi \to \frac{\phi}{\sqrt{\lambda}}}{\longrightarrow} \quad \frac{1}{\lambda}\int \left[\partial\bar{\phi}\partial\phi + \frac{1}{4}\left(\bar{\phi}\phi\right)^2\right] \equiv \frac{S}{\lambda}$$

$$\phi(x)^n \;\to\; \frac{1}{\lambda^{n/2}}\phi^n(x) \;\equiv\; \frac{1}{\lambda^{n/2}}\, e^{\frac{n\lambda}{\lambda}\,\ln\phi(x)}$$

$$\langle\bar{\phi}^n(x_f)\phi^n(x_i)\rangle \;=\; \frac{1}{\lambda^n}\,\frac{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, e^{-\frac{1}{\lambda}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 - \lambda n(\ln\bar{\phi}(x_f) + \ln\phi(x_i))\right]}}{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, e^{-\frac{1}{\lambda}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2\right]}}$$

Compute semiclassically for $\left[\begin{array}{l} \lambda \ll 1 \\ \lambda n = \text{fixed} \end{array}\right.$

saddle point $\phi_{cl} \equiv \phi_{cl}(\lambda n, x_{fi})$ $x_{fi} \equiv x_f - x_i$

$$\frac{1}{\lambda^n} \frac{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi \, e^{-\frac{1}{\lambda}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 - \lambda n \ln \bar{\phi}_f \phi_i\right]}}{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi \, e^{-\frac{1}{\lambda}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2\right]}}$$

$$= \frac{1}{\lambda^{n+\frac{1}{2}}} \, e^{\frac{1}{\lambda}[\Gamma_{-1}(n\lambda, x_{fi}) + \lambda\Gamma_0(n\lambda, x_{fi}) + \dots]}$$

↑ $U(1)$ breaking zero mode of solution

$$= n! \, e^{\frac{1}{\lambda}\left[\tilde{\Gamma}_{-1}(n\lambda, x_{fi}) + \lambda\tilde{\Gamma}_0(n\lambda, x_{fi}) + \dots\right]}$$

↑ Stirling

$$n!\, e^{\frac{1}{\lambda}\left[\tilde{\Gamma}_{-1}(n\lambda, x_{fi}) + \lambda \tilde{\Gamma}_0(n\lambda, x_{fi}) + \ldots\right]}$$

$$\frac{\gamma_n}{n} = P_0(\lambda n) + \lambda P_1(\lambda n) + \ldots$$

- proves ‘exponentiation’ to all orders
- semiclassical expansion valid for all λn as long as $\lambda \ll 1$
- must match diagrammatic expansion at $\lambda n \ll 1$!

main problem: finding classical solution

$$\partial^2\phi(x) - \frac{1}{2}\phi^2(x)\bar{\phi}(x) = -\frac{\lambda n}{\bar{\phi}(x_f)}\delta^{(d)}(x - x_f),$$
$$\partial^2\bar{\phi}(x) - \frac{1}{2}\phi(x)\bar{\phi}^2(x) = -\frac{\lambda n}{\phi(x_i)}\delta^{(d)}(x - x_i)$$

can perform perturbation theory in λn

not straightforward to find solution at finite λn

basic difficulty

must regulate to $d = 4 - \epsilon$ where ϕ^4 not scale invariant: radial dependence non-trivial

Way out

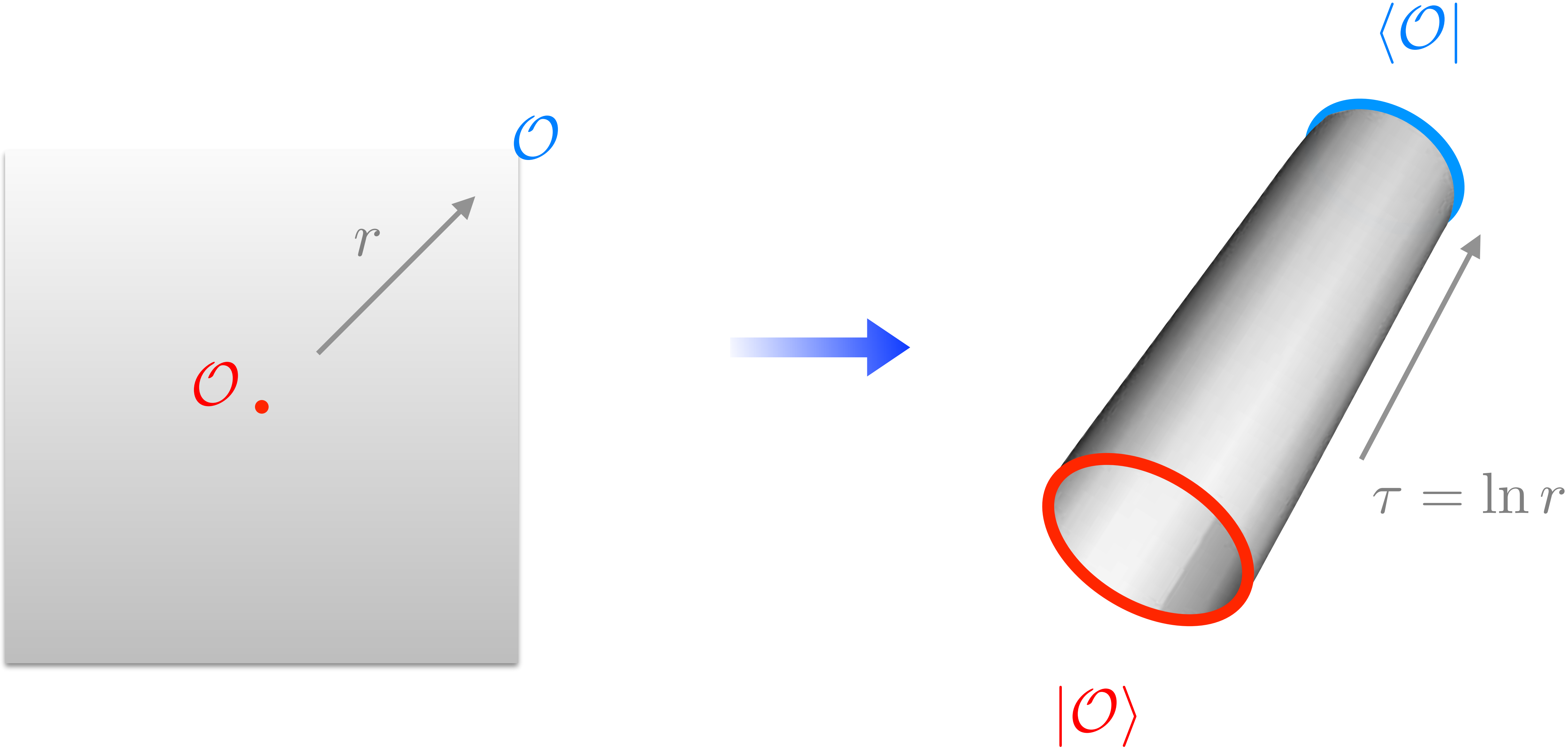
▲ focus on Wilson-Fisher fixed point in $d = 4 - \epsilon$

$$\beta_\lambda = \lambda \left[-\epsilon + 5\frac{\lambda}{(4\pi)^2} - 15\frac{\lambda^2}{(4\pi)^4} + O(\lambda^3) \right]$$

$$\beta(\lambda_*) = 0 \qquad \frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3\epsilon^2}{25} + O(\epsilon^3)$$

▲ Conformally map theory to cylinder $\mathbb{R} \times S_{d-1}$

Mapping to the cylinder & operator state correspondence



$$\langle\, \mathcal{O}(r)\, \mathcal{O}(0)\, \rangle = \frac{1}{r^{2\Delta}} \quad \longrightarrow \quad \langle \mathcal{O}| e^{-H\tau} |\mathcal{O}\rangle = e^{-\Delta\tau}$$

- on cylinder $D = H_{cy\ell}$ is conserved off-criticality

- expect solution to be stationary for $\tau_f \gg \tau_i$

- simple consistent ansatz at $\tau_f \gg \tau \gg \tau_i$

$$\phi_{c\ell} = \rho e^{i\chi} \qquad \begin{aligned} \rho &= \text{const} \\ \chi &= -i\mu\tau \end{aligned}$$

this corresponds to a homogeneous superfluid

$\mu \equiv$ chemical potential

$\phi^n \equiv$ lowest dimension primary of charge n, dominates path integral at large times in charge n sector

$$\phi \equiv \rho\, e^{i\chi}$$

projects on charge n

$$S_{cyl} = \int d\tau d\Omega_{d-1} \left(|\partial\phi|^2 + \xi_d \mathcal{R} |\phi|^2 + \frac{\lambda}{4}|\phi|^4 - i\frac{n}{\Omega_{d-1}}\chi_f + i\frac{n}{\Omega_{d-1}}\chi_i \right)$$

$$\xi_d \mathcal{R} = \left(\frac{d-2}{2}\right)^2 \equiv m_d^2$$

'conformal mass' on sphere

boundary eom
$$2\rho^2\mu = \frac{\lambda n}{S_{d-1}}$$

$$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

bulk eom
$$-\mu^2 + m_d^2 + \frac{1}{2}\rho^2 = 0$$

$$m_d^2 \equiv \left(\frac{d-2}{2}\right)^2$$

$$\mu(\mu^2 - m_d^2) = \frac{\lambda n}{4S_{d-1}}$$

Plug back into action and perform systematic loop expansion

$$\Delta_{\phi^n} = \frac{1}{\lambda_*}\Delta_{-1}(\lambda_* n) + \Delta_0(\lambda_* n) + \lambda_* \Delta_1(\lambda_* n) + \dots .$$

Leading order $\left[\lambda_* \propto \epsilon \to 0 \qquad \lambda_* n = \text{fixed}\right]$

$$\frac{\Delta_{\phi^n}}{n} = \frac{1}{\lambda_* n}\Delta_{-1}(\lambda_* n) + \frac{\lambda_*}{\lambda_* n}\Delta_0(\lambda_* n) + \ldots \quad \longrightarrow \quad \frac{1}{\lambda_* n}\Delta_{-1}(\lambda_* n)$$

$$\frac{1}{\lambda_* n}\Delta_{-1} = \frac{3\left[9x - \sqrt{81x^2 - 3}\right]^{1/3} + 3^{2/3}\left[9x - \sqrt{81x^2 - 3}\right]}{\left[\left(9x - \sqrt{81x^2 - 3}\right)^{2/3} + 3^{1/3}\right]^2}$$

$$+ \frac{9 \times 3^{1/3} x \left[9x - \sqrt{81x^2 - 3}\right]^{2/3}}{2\left[\left(9x - \sqrt{81x^2 - 3}\right)^{2/3} + 3^{1/3}\right]^2} \qquad x \equiv \frac{\lambda_* n}{16\pi^2}$$

Supposed to resum leading powers of n at all loops!

expanding at small λn

$$\Delta_{\phi^n} = n + \frac{\lambda n^2}{32\pi^2} - \frac{\lambda^2 n^3}{512\pi^4} + \frac{\lambda^3 n^4}{4096\pi^6} + O\left(\lambda^4 n^5\right)$$

and comparing with diagrams

$$\gamma_n = \frac{\lambda n(n-1)}{32\pi^2} - \frac{\lambda^2 n^2(n-1)}{512\pi^4} + \ldots$$

they happily agree

Makes one want to check subleading terms

To do so fixed point condition in $d = 4 - \epsilon$ is essential

diagrammatic computation gives

$$\gamma_n\Big|_{\text{diag}} = \epsilon\frac{n(n-1)}{10} - \epsilon^2\frac{n(n^2-4n)}{50} + O\left(\epsilon^3 n^4\right)$$

Semiclassically, subleading terms should come from Casimir energy on cylinder

Semiclassically on cylinder

1-loop

$$e^{-\Delta_n \tau} = \frac{\int D\phi \, e^{-S+in\chi_f - in\chi_i}}{\int D\phi \, e^{-S}} = \frac{e^{-S_{cl}(n) - \frac{1}{2}\ln\det S_n^{(2)}}}{e^{-\frac{1}{2}\ln\det S_0^{(2)}}}$$

$$\frac{1}{\tau}\left(\frac{1}{2}\ln\det S_n^{(2)} - \frac{1}{2}\ln\det S_0^{(2)}\right) = \frac{1}{2}\sum_{\ell} n_{\ell,d}\left[\omega_+(\ell,d) + \omega_-(\ell,d) - 2\omega_0(\ell,d)\right]$$

Casimir Energy on S_{d-1}, convergent for sufficiently low d

in terms of the renormalized coupling λ_R

$$\gamma_n = \frac{\lambda_R n(n-1)}{32\pi^2} + \frac{\lambda_R \epsilon n^2}{64\pi^2} - \frac{\lambda_R^2(2n^3+3n^2)}{1024\pi^4} + \dots$$

fixed point

$$\epsilon\frac{n(n-1)}{10} - \epsilon^2\frac{n(n^2-4n)}{50} + O\left(\epsilon^3 n^4\right)$$

perfect match !

matching to available computations with n ≤ 4 up to 5-loops

$$\gamma_{\phi^n} = n \sum_{\ell=1} \varepsilon^\ell P_\ell(n)$$

$$P_3(n) = \frac{n^3}{125} + \frac{n^2[16\zeta(3) - 29]}{500} + \frac{n[599 - 672\zeta(3)]}{5000} + \frac{[1024\zeta(3) - 603]}{10000}, \quad (91)$$

$$P_4(n) = -\frac{21n^4}{5000} + \frac{n^3[214 - 77\zeta(3) - 80\zeta(5)]}{5000} + \frac{n^2[66336\zeta(3) + 160\pi^4 - 89491]}{600000} + \frac{n[41073 - 45864\zeta(3) + 46720\zeta(5) - 224\pi^4]}{200000} + \frac{75888\zeta(3) - 130560\zeta(5) + 512\pi^4 - 53717}{600000}, \quad (92)$$

$$P_5(n) = \frac{n^5 8}{3125} + \frac{n^4[476\zeta(3) + 480\zeta(5) + 448\zeta(7) - 1683]}{50000} + 0.00093n^3 - 0.01067n^2 - 0.2460n + 0.2680. \quad (93)$$

total match at 3-loops, partial match at 4-loops plus boosting of available computations

Large charge limit for $d \to 4$

$$\left[\begin{array}{l}\lambda = \lambda_* \propto \epsilon \to 0 \\ \lambda n \gg 1\end{array}\right]$$

$$\Delta_{\phi^n} = \frac{\pi^2}{\lambda}\left[\frac{3}{8}\left(\frac{\lambda n}{\pi^2}\right)^{4/3} + \left(\frac{\lambda n}{\pi^2}\right)^{2/3} - \frac{2}{3} + O\left(\left(\lambda n/\pi^2\right)^{-2/3}\right)\right]$$

Interpretation: $m_\rho^2 \sim \mu^2 \sim (\lambda n)^{2/3} \gg \frac{1}{R^2} = 1$

→ integrate out radial mode: 'pure' conformal superfluid EFT

$$\mathcal{L} \sim (\partial\chi)^4 + \mathcal{R}(\partial\chi)^2 + \mathcal{R}^2 + \ldots$$

known large charge behaviour

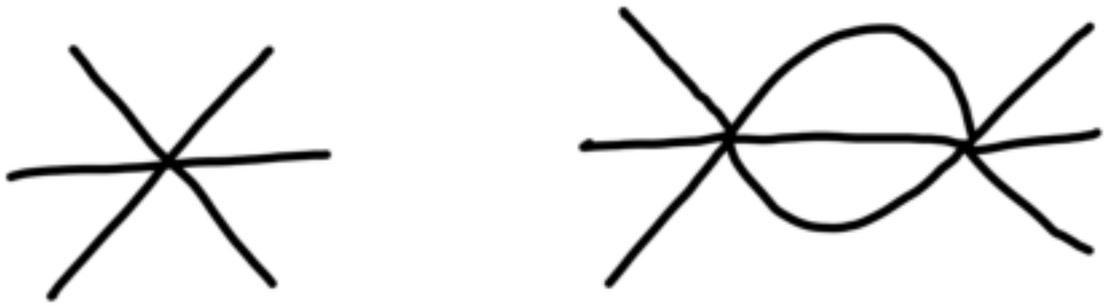
Hellerman, Orlando, Reffert, Watanabe '15
Monin, Pirtskhalava, RR, Seibold '16
Jafferis, Mukhametzhanov, Zhiboedov, '17

For $d = 4 - \epsilon$ quantum corrections provide proper scaling

$$\Delta_{\phi^n} = \frac{1}{\epsilon}\left[c_{-1}(\epsilon)\left(\frac{2}{5}\epsilon n\right)^{\frac{4-\epsilon}{3-\epsilon}} + c_0(\epsilon)\left(\frac{2}{5}\epsilon n\right)^{\frac{2-\epsilon}{3-\epsilon}} + \dots\right]$$

	$c_{-1}(1)$	$c_0(1)$
Monte-Carlo	0.337(3)	0.27(4)
ε-expansion: LO	0.47	0.79
ε-expansion: NLO	0.42	0.04

$$\frac{1}{\lambda}\left(|\partial\phi|^2 + |\phi|^6\right) \qquad \textbf{in} \qquad 3-\epsilon$$



$$\beta(\lambda) = \epsilon\lambda + a\lambda^3$$

in d=3 conformally invariant for any λ up to 1-loop

$$\Delta_n = \underbrace{\frac{1}{\lambda} f_0(\lambda n, \epsilon) + f_1(\lambda n, \epsilon)}_{\epsilon \to 0} + \lambda f_2(\lambda n, \epsilon) + \ldots$$

Result non-trivially interpolates between universal quantum effects in genuine 3D CFT at large charge and Feynman diagram computations

Summary

Wilson-Fisher fixed points: simple but rich playground to get structural insight on $\lambda n \gg 1$ regime in QFT

Loop expansion for γ_{ϕ^n} non-trivially and systematically encapsulated by semiclassical supefluid configuration

Properties of nearby operators, ex $\phi^{n-2}\partial\phi\partial\phi$, described by hydrodynamic modes

$\langle\phi^{n_1}\ldots\phi^{n_p}\rangle$ can be studied by extension of method, providing dynamical information, akin to amplitudes

...but it would be nice to get back to the SM... someone certainly will before the FCC begins