Berry phase in quantum field theory

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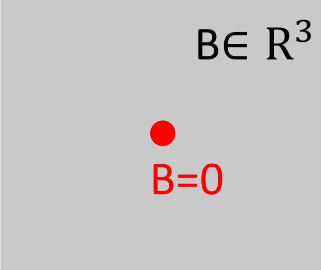
Diabolical point in quantum mechanics



- Spin ½ in magnetic field B: two eigenstates with energy $\pm\mu|B|$

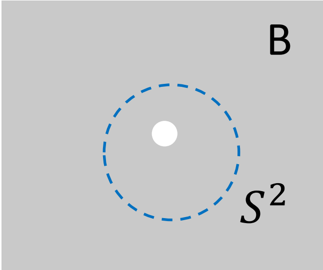
$$H(B) = \mu\sigma \cdot B, \qquad B \in \mathrm{R}^3$$

- $B = 0$: degenerate zero energy states, codimension 3 in the parameter space R^3.
- $B \neq 0$: unique ground state with energy gap (i.e. trivially gapped)
- The point $B = 0$ has ground state differed from the surrounding region: "diabolical point" in the parameter space. [Berry]
- For generic Hamiltonian without symmetry, level-crossing occurs in codimension 3 (H spanned by 3 Pauli matrices) [von Neumann, Wigner]

Describe “diabolical point” using Berry phase

- Removing diabolical points from parameter space creates non-trivial topology $\mathrm{R}^3\backslash 0 \sim S^2$. Detect diabolical point by the topology?
- Ground state wavefunction does not depend continuously on $S^2_{(\theta,\phi)}$: $|0\rangle_N = \left(\sin\frac{\theta}{2}e^{i\phi}, -\cos\frac{\theta}{2}\right), |0\rangle_S = \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}e^{-i\phi}\right)$. Transition function at equator $|0\rangle_N = e^{i\phi}|0\rangle_S$

[Berry], [Simon]



Berry connection $A_B = i\langle 0|\partial_B|0\rangle$ with non-trivial Berry phase

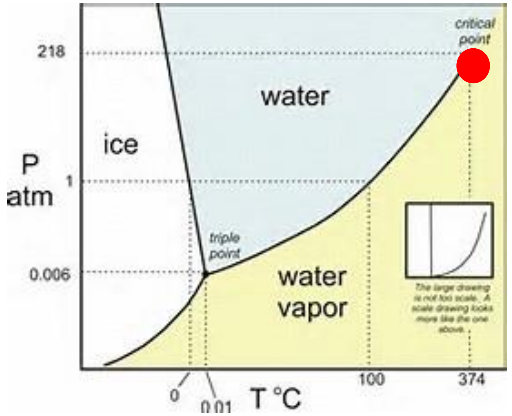
$$\text{Berry number:} \quad \oint_{S^2} F_B/2\pi = 1$$

Berry curvature $F_B = dA_B$ has quantized period, has singularity at diabolical point

- Berry number $\neq 0 \Rightarrow$ not trivially gapped everywhere : if the theory were trivially gapped in R^3, S^2 would be contractible and the Berry number would be zero.

Diabolical Points in Many Body Systems (Outline)

- Critical points often sit at the end of first order phase transition lines



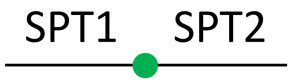
Critical point connected to 1^{st} order phase transition line

- There are also isolated critical points (diabolical points) in phase diagram surrounded by gapped phase with non-degenerate vacuum.

Q1: What protects the stability of such diabolical critical points?

Diabolic Points in Many Body Systems (Outline)

- Usual argument for stability: symmetry protected topological (SPT) phase protects against symmetry preserving perturbations.



What about symmetry-breaking perturbations or systems without symmetry?

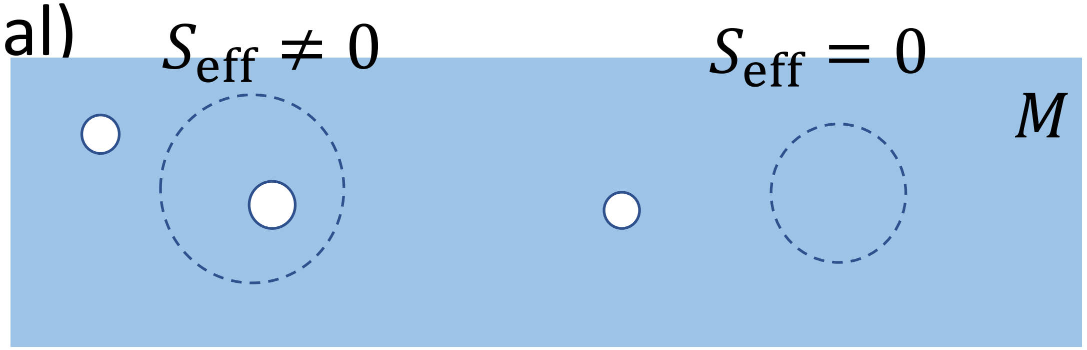
- Ans: Berry phase. Generalization of SPT.
- Q2: Does Berry phase have bulk-boundary correspondence? (Yes)
- Q3: What's the classification of Berry phases? (Cobordism group)
- Applications: use Berry phase to study Néel / Valence Bond Solid transition, and new tests for infrared Chern-Simons matter dualities

Berry Phase Protects Phase Transitions

- Parameter space M consists of trivially gapped region (blue) with possible diabolical loci (white) removed. Create non-trivial topology
- Promote the parameters to be position-dependent background fields

$$\phi\text{: spacetime} \rightarrow M$$

- Non-trivial topological term in effective action $S_{\text{eff}}[\phi]$ for configuration $\phi(x) \in S^r \subset M$ protects diabolical loci inside. (otherwise sphere would be contractible in M and the topological term is trivial)



Effective action in quantum mechanics

- Let the parameter (magnetic field) to be periodic in time

$$B\colon S^1_{\text{time}} \to \mathrm{R}^3$$

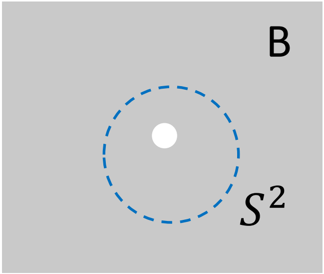
- Away from diabolical point (nonzero magnetic field), $\mathrm{R}^3\backslash 0 = S^2$: topological effective action

$$S_{\text{eff}} = \int B^* \tau_1$$

τ_1 is a 1-form on S^2 i.e. the Berry connection. Adiabatic $|0\rangle \to |0\rangle e^{iS_{\text{eff}}}$

τ_1 undergoes gauge transformation across coordinate patches on S^2 .

$e^{iS_{\text{eff}}}$ is gauge invariant, but the effective Lagrangian is not



Effective action in quantum mechanics as Wess-Zumino-Witten term

- We can also write the effective action using the gauge invariant Berry curvature.
- Introduce one-parameter family of background $B(t,s)$ defined on 2-manifold Y that bounds S^1_{time}

$$S_{\text{eff}} = \int_Y d(B^* \tau_1) = \int_Y B^* H_2$$

- Berry curvature $H_2 = d\tau_1$ is a 2-form on S^2 with quantized period
- The action is an example of Wess-Zumino Witten (WZW) term. It is characterized by non-trivial map $S^2_{(t,s)} \to S^2 \subset \mathrm{R}^3$ whose degree is the Berry number $\pi_2(S^2) = \mathrm{Z}$.

Effective action in (d+1) Dimension Spacetime

- Promote parameters to be spacetime-dependent background field

$$\phi: X_{d+1}^{\text{spacetime}} \to M$$

Trivially gapped with diabolical loci removed, which creates topology.

The Berry phase is the topological term in effective action

$$S_{\text{eff}}[\phi] = \int \phi^* \tau_{d+1}$$

where τ_{d+1} is a $(d+1)$-form on parameter space M. $\tau_{d+1} \to \tau_{d+1} + d\lambda_d$

- $S_{\text{eff}}[\phi] = \int_{Y_{d+2}} \phi^* H_{d+2}$ for Y_{d+2} bounds spacetime and $H_{d+2} = d\tau_{d+1}$ has quantized period. Such WZW term can arise from $\pi_{d+2}(M)$.
- Family of lattice Hamiltonian systems: Berry phase is studied in [Kapustin, Spodyneiko]1

Include global symmetry (generalization of SPT phase)

- $S_{\text{eff}}[\phi, A]$ with background gauge field A for the symmetry
- Example: Thouless pump for $U(1)$ symmetry

$$S_{\text{eff}}[\phi, A] = \int_{X_{d+1}} A \wedge \phi^* \tau_d \,, \quad \tau_d\text{: closed } d\text{-form on parameter space}$$

- $U(1)$ current for spacetime-dependent parameter: charge associated with soliton configuration of ϕ

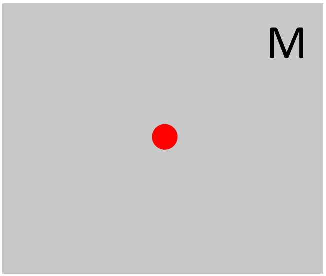
$$j = \star\, (\phi^* \tau_d), \qquad Q = \oint_{S^d} \star\, j = \oint_{S^d} \phi^* \tau_d$$

- Family of lattice Hamiltonian systems with $U(1)$ symmetry: Thouless pump invariant studied in [Kapustin, Spodyneiko]2
- We will give example of diabolical points protected by higher-degree currents generating higher-form symmetry [Gaiotto,Kapustin,Seiberg,Willett],[Wen]

Example: Free Fermions in (1+1)d

- One Dirac fermion with complex mass $M = me^{i\alpha} \in \mathrm{R}^2$

$$m\bar{\psi}e^{i\gamma_{01}\alpha}\psi = M\bar{\psi}_+\psi_- + \mathrm{h.c.}$$



The theory has $U(1)_V$ symmetry $\psi \to U\psi$.

- $M \neq 0$: gapped with a unique ground state.
- $M = 0$: codimension 2 gapless diabolic point.
- $M = 0$: mixed anomaly for $U(1)_A - U(1)_V$, but $U(1)_A$ absent for $M \neq 0$

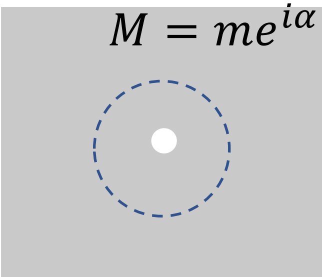
Q: Can we add interaction to gap out the system preserving only $U(1)_V$ symmetry? (No. Diabolic point protected by Thouless pump invariant.)

Q: Is there family of trivially gapped interfaces depend continuously on the parameters? (No, example of bulk-boundary correspondence for Berry phase)

Diabolic Point Protected by Thouless Pump

- Denote the $U(1)_V$ background gauge field by A.

For $|M| > 0$ the effective action for massive fermion has the Thouless pump invariant that protects the diabolic point [Goldstone, Wilczek]



$$S_{\text{eff}} = \int A \frac{d\alpha}{2\pi} = \frac{1}{2\pi}\int \alpha dA = \int A \star j$$

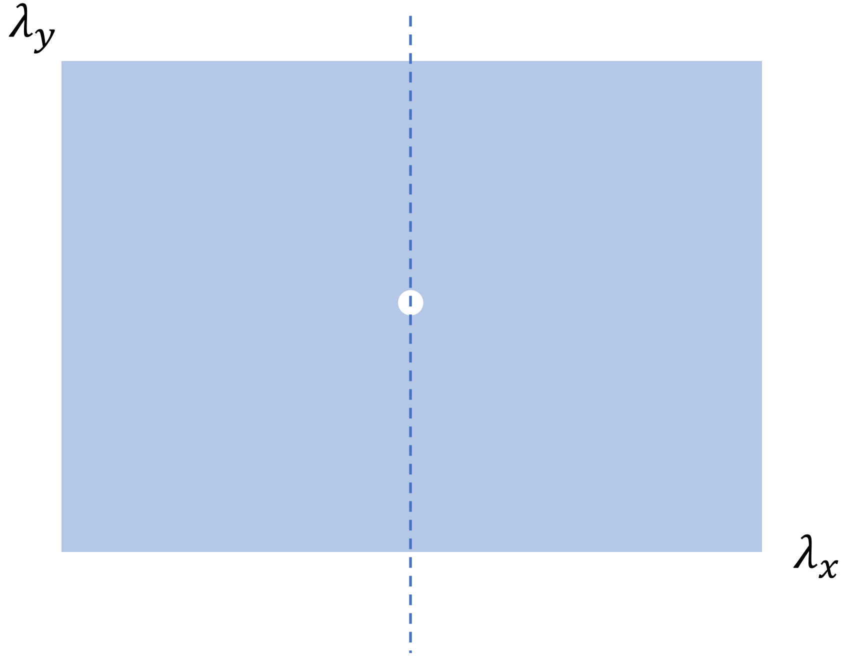
$\frac{1}{2\pi}\oint d\alpha = \frac{1}{2\pi}\oint d\arg M = 1$: the loop is not contractible. The gapless point at the origin cannot be completely removed.

- $U(1)_V$ current, $j = \frac{1}{2\pi}\star d\alpha$. For $\alpha = \alpha(t)$ periodic in time and winds origin N times: pumps charge $\Delta Q = \oint \star j = \frac{1}{2\pi}\big(\alpha(T) - \alpha(0)\big) = N$.

[Thouless]

Perturbation cannot remove the diabolical point

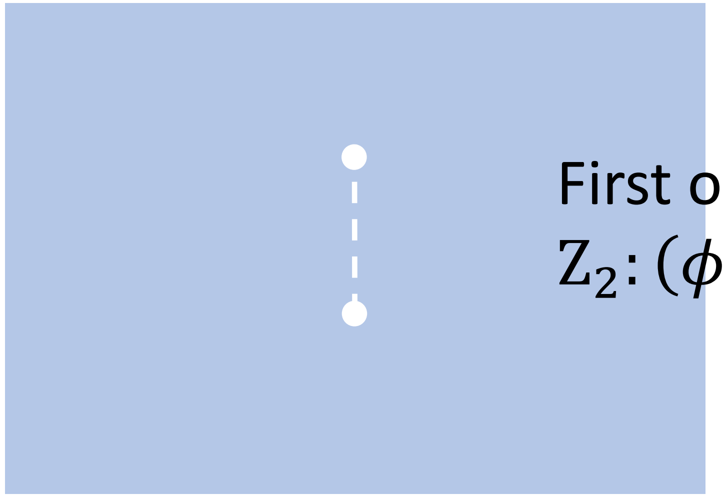
- Consider bosonization description of the free fermion in (1+1)d with periodic scalars ϕ, θ. $U(1)_V$ symmetry $\theta \to \theta + \alpha$.
- Fermion mass corresponds to $\lambda_x \sin\phi + \lambda_y \cos\phi, M = \lambda_x + i\lambda_y$



$\lambda_x = 0$: charge conjugation symmetry Z_2: $(\phi, \theta) \to (-\phi, -\theta)$

Perturbation cannot remove the diabolical point

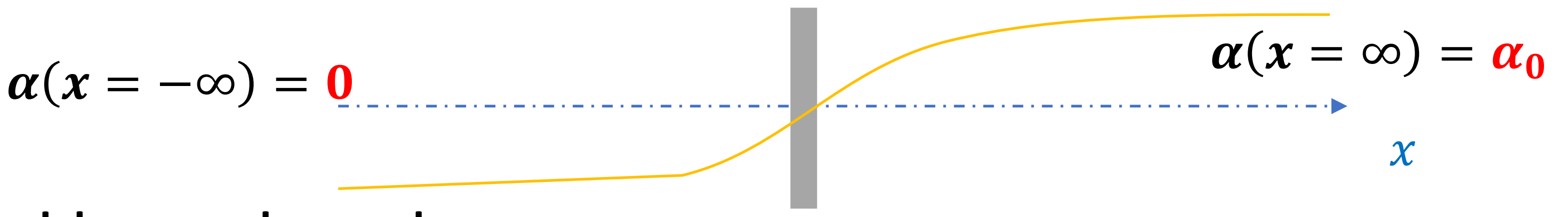
- Decrease radius pass through the self-dual point, new relevant perturbations $\cos 2\phi$, $\sin 2\phi$. Deformation by $\cos 2\phi$ changes the phase diagram



First order phase transition: broken $Z_2: (\phi, \theta) \rightarrow (-\phi, -\theta)$

- Perturbation can only deform the diabolical point to be diabolical loci, and it persists in the phase diagram

Family of Interfaces



- Interface labelled by α_0 has charge

$$\Delta q = \int \star j = \int d\alpha/2\pi = \alpha_0/2\pi\,, \qquad (\Delta q = 1 \text{ for } \alpha_0 = 2\pi)$$

- For $\alpha(x) = \alpha_0\theta(x)$, bound state $\psi = \psi(0)e^{-\beta|x|}$ for $\beta > 0$. $\beta = m\sin(\alpha_0/2)$, energy $E = m\cos(\alpha_0/2)$

Single normalizable zero mode at $\alpha_0 = \pi$ [Jackiw, Rebbi]

Non-normalizable mode at $\alpha_0 = 0,2\pi$, merges with the bulk modes

- The existence of single zero mode and non-normalizable modes can also be found in smooth interfaces in this theory with Thouless pump

[Keil,Kobes]

Boundary-Bulk Correspondence for interface

- The family of interfaces cannot be described by purely (0+1)d quantum mechanics for all α_0. Suppose otherwise, effective action

$$\int q(\alpha_0) A_t dt$$

$q(\alpha_0) \in \mathrm{Z}$ is vacuum charge: jump only at gapless points and single-valued (sum of all jumps when α_0 varies from 0 to 2π is $\Delta q = 0$)

- Thouless pump: $\Delta q = 1$ when α_0 varies from 0 to 2π. Contradiction.
- Reason: q can also jump at delocalized point. Δq from interface gapless points = $-\Delta q$ from non-normalizable modes escaped to bulk spatial infinity (Thouless pump)

General Bulk-Boundary Correspondence

- Consider family of bulk with Berry phase in (d+1) dimension. Namely, the phase diagram has diabolical loci but otherwise trivially gapped with nontrivial topological term in the effective action
- If bulk system with Berry phase has a boundary, then the gap must close on the boundary for some parameter. This arises at boundary diabolical points (loci).
- Boundary-bulk ``Anomaly inflow'':

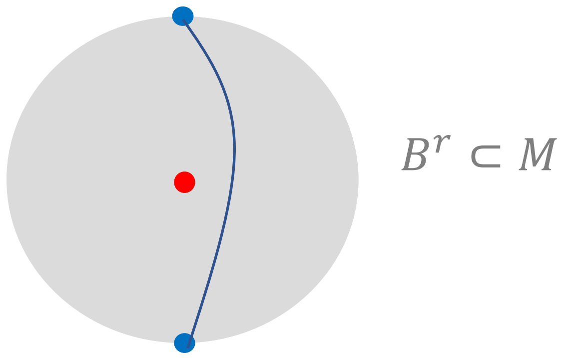
Total Berry number of boundary diabolical points

= Total Berry number in the bulk.

(Similar to the Nielsen-Ninomiya theorem)

General Bulk-Boundary Correspondence

- Consider a diabolic point (red) enclosed by the ball $B^r \subset M$ described by bulk Berry phase.
- Consider a family of interfaces (special case: boundary) where the parameter interpolates between the North and South pole. The interpolation is a curve in B^r connecting the two points
- The interface whose curve hits the red point has diabolic point on the interface



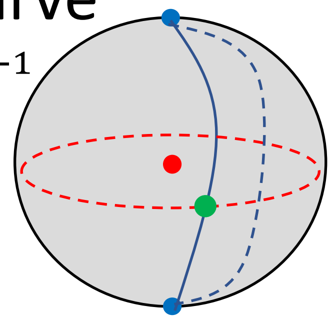
General Bulk-Boundary Correspondence

- What happens when the family of interfaces varies away from the diabolic point? Restrict curves to lie on S^{r-1}

Bulk Berry phase implies that as the family of interfaces swipes around S^{r-1} there is additional phase, so the family cannot depend continuously on the parameters

- Represent the parameter on interface as intersection (green) of the curve with the equator

$$M_{\text{interface}} = S^{r-2}$$



- Compute interface Berry phase: introduce 1-parameter family of background to count how many times green dot swipes around S^{r-2}.

As green dot swipes S^{r-2} once, the blue curve swipes around S^{r-1} once

Boundary Berry number = Bulk Berry number

Example: particle on a circle

- Particle on circle $x \sim x + 2\pi$ with $U(1)$ symmetry and parameter α
$$\frac{1}{2}\dot{x}^2 + \frac{1}{2\pi}\alpha\dot{x}, \qquad \alpha \sim \alpha + 2\pi, \qquad U(1)\colon x \to x + f$$
- Background $U(1)$ gauge field A: $\frac{1}{2}(\dot{x} - A)^2 + \frac{1}{2\pi}\alpha(\dot{x} - A)$
- α is no longer periodic, violated by $\int A$. ``an anomaly'' [Córdova,Freed,Lam,Seiberg]
- The theory lives on the boundary of the bulk with Berry phase
$$S_{\text{eff}}^{\text{bulk}} = \int A \frac{d\alpha}{2\pi}$$
- Bulk-boundary correspondence: level-crossing occurs on boundary at some α (in fact, $\alpha = \pi$)

Berry phase v.s. Anomaly: Two Free Dirac fermions in (1+1)d

- Mass $\bar{\psi}\left(M_0 + i\gamma^{01} M_i \sigma^i\right)\psi,\ M_A = (M_0, M_1, M_2, M_3) \in \mathrm{R}^4$

$|M| > 0$: gapped with a unique ground state. $\mathrm{R}^4 - \{0\} \sim S^3$

$|M| = 0$ codimension-4 gapless diabolical point.

- Gapless point $|M| = 0$ is protected by 't Hooft anomaly for $Spin(4)$ symmetry only against symmetry-preserving perturbation
- Gapless point also protected by Berry phase, also against symmetry-breaking perturbations
- Effective action: one parameter family of background $Y_{(t,x,s)}$ with Wess-Zumino term $\pi_3\left(S^3\right) = \mathrm{Z}$. $S_{\mathrm{eff}} = \int \omega_2$, $H = \mathrm{d}\omega_2$ the volume form of S^3

$$H = 1/(6\pi|M|^4)\ \epsilon^{ABCD} M_A dM_B dM_C dM_D$$

[Abanov, Wiegman]

Free Fermion in (2+1)d: No Perturbative Anomaly

- Two Dirac fermions with mass term

$$m n_i \bar{\psi} \sigma^i \psi, \qquad n_i \in S^2, \sum n_i^2 = 1$$

Gapped for $m > 0$, gapless for $m = 0$ (diabolical point of codimension 3)

- $m = 0$ has $SU(2)$ symmetry, protected by mixed parity- $SU(2)$ anomaly

Also protected by Thouless pump invariant: skyrmion current $\pi_2(S^2) = \mathrm{Z}$

$$S_{\text{eff}}[m n_i, A] = \int A_\mu j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\rho} \epsilon^{ijk} \int A_\mu n_i \partial_\nu n_j \partial_\rho n_k$$ [Abanov, Wiegman]

- Berry phase has infinite order i.e. remains nontrivial in any N-copy systems

Parity anomaly has order 2: no anomaly by taking 2 copies, only robust against $SU(2)$ sym preserving perturbation

- In addition, the effective action also has $\theta = \pi$ Hopf term $\pi_3(S^2) = \mathrm{Z}$

Web of diabolical points related by gauging U(1) symmetry

- Starting from a system with diabolic loci protected by Thouless pump invariant, can we obtain new system protected by Thouless pump invariant? Gauging the $U(1)$ symmetry [Kapustin, Strassler], [Witten]

$$S_{\text{eff}}[\phi, A] \to \text{gauging } U(1) \to S_{\text{eff}}^{\text{new}}[\phi, B] = S_{\text{eff}}[\phi, a] + \frac{1}{4\pi} ada - \frac{1}{2\pi} adB$$

New $U(1)$ magnetic symmetry $j = \star\, da/2\pi$.

$$S_{\text{eff}}[\phi, A] = \int A\phi^* \tau_2\,, \qquad S_{\text{eff}}^{\text{new}}[\phi, B] = \int B\phi^* \tau_2 + H[\phi] - \frac{1}{4\pi} BdB$$

- Two free Dirac fermions in (2+1)d ⇒ Interacting $U(1)_1$ with two fermions (realizes phase transition betw S^1 sigma model and $U(1)_2$) protected by the same Thouless pump invariant

$U(1)$ Gauge Theory with 2 Scalars in (2+1)d

- $U(1)$ gauge theory with two Wilson-Fisher scalars deformed by $SU(2)$ triplet mass $m^2 n_i \in \mathrm{R}^3$

$$V(\phi) = m^2 n_i \phi^+ \sigma^i \phi + \lambda(\phi^+\phi)^2$$

- $m \neq 0$: $SU(2)$ symmetry explicitly broken. $U(1)$ gauge field is Higgsed

$$a = \frac{n_1 dn_2 - n_2 dn_1}{2(1-n_3)} \Rightarrow da = -\frac{1}{2}\epsilon^{ijk} n_i dn_j dn_k$$

Monopole ⇒ skyrmion $\pi_2(S^2) = \mathrm{Z}$

The $U(1)$ magnetic symmetry $j = -\frac{1}{2\pi} \star da$ has Thouless pump

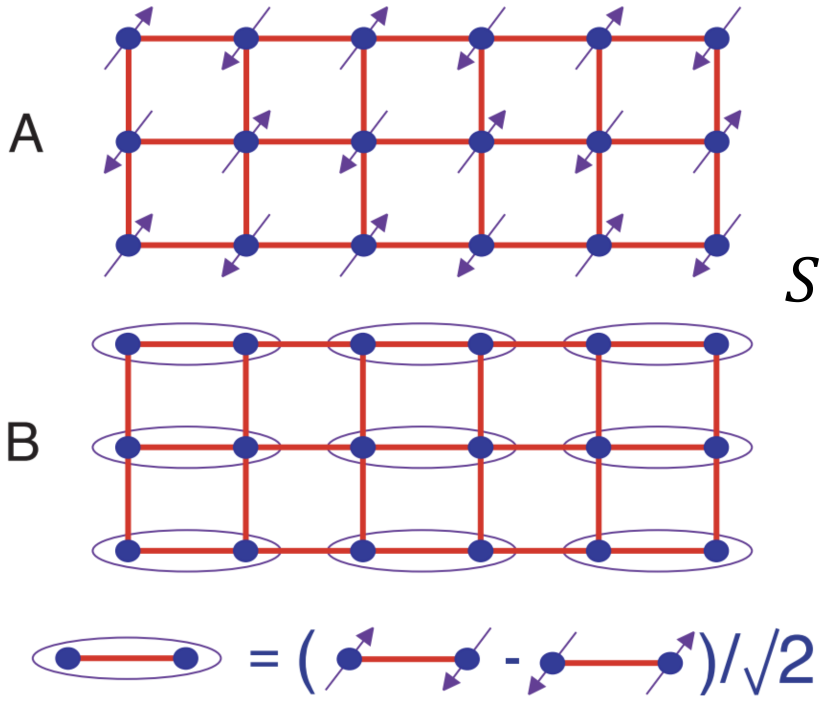
$$S_{\mathrm{eff}}[m^2 n_i, A] = \frac{1}{8\pi}\epsilon^{\mu\nu\rho}\epsilon^{ijk}\int A_\mu n_i \partial_\nu n_j \partial_\rho n_k$$

The phase transition $m = 0$ is protected by the Thouless pump invariant

Application: Deconfined Quantum Criticality

- Deconfined quantum critical point: scalar QED3 without Chern-Simons term is believed to describe a deconfined quantum critical point between the Néel state and the valence bound solid (VBS).

[Senthil,Vishwanath,Balents,Sachdev,Fisher]



Néel: broken $SO(3)$ spin symmetry

$SO(3)$ vector N_i Néel field $\sim SO(3)$ vector mass $m^2 n_i$

VBS: broken Z_4 lattice rotation symmetry

$U(1)$ with two scalars: $SO(3) \times U(1)$ symmetry

Application: Deconfined Quantum Criticality

- The transition is protected by ‘t Hooft anomaly for $SO(3)\times U(1)$ symmetry against symmetry preserving perturbations.

  [Benini,Hsin,Seiberg],[Wang,Nahum,Metliski,Xu,Senthil],[Komargodski,Sharon,Thorngren,Zhou]……

- Symmetry of the action is

$$\left(U(1)_{\mathrm{gauge}}\times SU(2)_{\mathrm{global}}\right)/\mathrm{Z}_2$$

[Benini,Hsin,Seiberg]

Faithful flavor symmetry is $SO(3)$

$SO(3)$ bundles that are not $SU(2)$ bundles: changes the quantization of the flux of $U(1)_{\mathrm{gauge}}$. Anomaly for the $U(1)$ magnetic symmetry

$$\pi\int\frac{dA}{2\pi}w_2(SO(3))$$

Application: Deconfined Quantum Criticality

- The anomaly has order 2
- For lattice that does not respect $\mathrm{Z}_2 \subset U(1)$ symmetry: the anomaly vanishes and does not offer protection to the phase transition.
  - For honeycomb lattice with Z_3 symmetry there could be intermediate trivially gapped phase [Jian, Zaletel], [Po,Watanabe,Jian,Zaletel]
- Here we show if there is a $\mathrm{Z}_N \subset U(1)$ symmetry for any $N \neq 1$ there is non-trivial transition protected by the Thouless pump invariant. Does not need $SO(3)$ symmetry. Protection even on honeycomb lattice where $N = 3$.

Generalization with Chern-Simons term and higher rank gauge group

- Chern-Simons matter theory: level k gives $\theta = k\pi$ Hopf term for the parameter field associated with $\pi_3(S^2) = \mathrm{Z}$ [Wilczek,Zee]
- Magnetic charge corresponds to Skyrmion number

$$\oint \frac{da}{2\pi} = -\frac{1}{4\pi} \epsilon^{ijk} \oint n_i dn_j dn_k$$

Skyrmion has spin $\frac{\theta}{2\pi} = \frac{k}{2}$: agrees with the spin of the monopole
[Wilczek,Zee]

- The discussion can be generalized to $U(N)$ gauge theory

Application: New Test for Boson/Fermion Duality

- $U(N)_1$ with Wilson-Fisher scalars are conjectured to flow to free Dirac fermions in the infrared [Hsin,Seiberg]

$$2\psi \leftrightarrow U(N)_1 + 2\phi, \qquad N \geq 2$$

- New consistency check: $SU(2)$ adjoint mass deformation on both sides

Constant mass: no information.

Novelty: promote mass to be spacetime-dependent.

- produces the same Berry phase effective action ($\theta = \pi$ Hopf term and the Thouless pump invariant)
- The effective action cannot be removed by adding local counterterm, must match across duality (not well-define at $m = 0$ where n_i are ill-defined)
- New test for web of dualities related by gauging or RG flow, e.g.

$$\psi \leftrightarrow U(1)_1 + \phi$$

[Seiberg,Senthil,Wang,Witten],[Karch, Tong]…

Thouless Pump with Higher-Form Symmetry

- Similar analysis applies to (3+1)d $U(1)$ gauge theory with 2 scalars
- The current $j = -\frac{1}{2\pi} \star da$ is a 2-form in (3+1)d: magnetic $U(1)$ 1-form symmetry that transforms the 't Hooft line operators.
- Same computation implies that the phase transition is protected by Thouless pump invariant

$$S_{\text{eff}}[m^2 n_i, A^{(2)}] = \frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} \epsilon^{ijk} \int A^{(2)}_{\mu\nu} n_i \partial_\rho n_j \partial_\sigma n_k$$

$A^{(2)}_{\mu\nu}$: background gauge field for the $U(1)$ 1-form symmetry

Further Examples: Thouless Pump with Gravitational Invariant

- Free Weyl fermion in (3+1)d with complex mass $M = me^{i\alpha}$

$|M| = m > 0$: gapped with a unique ground state

$|M| = 0$: codimension 2 gapless diabolical point

- Diabolical point protected by effective action for $m > 0$

$$S_{\text{eff}} = \frac{1}{384\pi^2}\int \alpha \, \mathrm{Tr}\, R^2 = -\frac{1}{2\pi}\int d\alpha \, \mathrm{CS}_{\text{grav}}$$

- The effective action cannot be removed by a well-defined function of M since $\alpha = \arg M$ is not well-defined at the origin. Berry curvature has singularity.
- This is called an ``anomaly'' in [Córdova,Freed,Lam,Seiberg]

Further Examples: Thouless Pump with Gravitational Invariant

$$S_{\text{eff}} = -\frac{1}{2\pi}\int d\alpha \, \text{CS}_{\text{grav}}$$

- For $\alpha = \arg M$ winds around the origin of x^1- x^2 plane n times, there will be gapless chiral edge mode propagating along x^3 with $c_- = \frac{n}{2}$.
- If turn on coupled to $U(1)$ gauge field, additional term $\frac{1}{8\pi^2}\int \alpha \, \text{Tr}\, F^2$

Classification of Berry Phase

- There is no stable diabolical locus (protected by Berry phase) without symmetry in codimension $m > d + 3$. Intuitively, for region B^m in the parameter space contains the diabolic point, $\pi_{d+2}\left(S^{m-1}\right) = 0$ if $m > d + 3$.
- In general we propose the following classification for Berry phase: family of trivially gapped theories parametrized by M are classified by

$$\Omega_s^{d+1}(M)$$

Where s represents additional structure (e.g. $SO, Spin$)

- Including symmetry G: replace the argument by $M \times BG$ possibly with twist (G action on M). If trivial M: known classification for SPT

[Kapustin], [Freed,Hopkins]

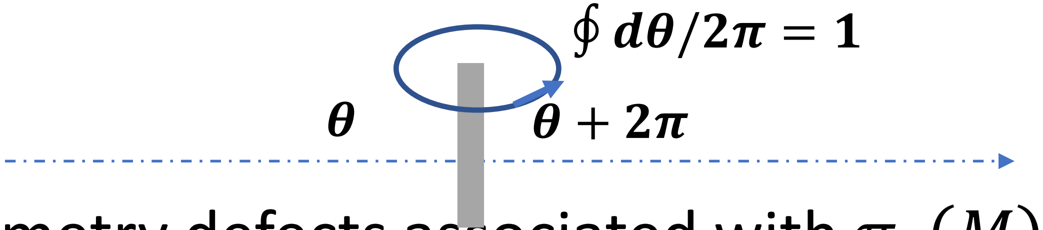
Berry phases of other degrees as Symmetries

- Phase diagram can have topology not captured by the Berry phase in (d+1) dimension. E.g. What about $\pi_k(M)$ for $k \neq d+2$?

They can define symmetry defects

- Example: 4d $N=1$ $\mathrm{SU}(n)$ gauge theory with θ angle, $\theta \sim \theta + 2\pi$

Define interface such that θ changes by 2π across the interface



0-form symmetry defects associated with $\pi_1(M) = \pi_1(S^1) = \mathrm{Z}$.

Symmetry permutes n vacua: spontaneously broken. ``vacuum-crossing''

[Sharon]

Berry phases of other degrees as Symmetries

- Similarly, $\pi_k(M)$ can define symmetry defect of various codimensions and they generate higher form symmetries. The symmetries may not act faithfully and may be spontaneously broken.
- It would be interesting to explore the implications of these symmetries.

Conclusion

- We use Berry phase to study diabolical points in phase diagram for system in general dimensions
- We argue that the theory with Berry phase implies the gap closes on the boundary for some parameter
- We discuss examples including free fermions and interacting gauge theory with bosons or fermions.
- Proposal for classifying family of invertible theories: cobordism group for the parameter space
- Applications include new evidence of the stability of the deconfined quantum critical point in Néel-VBS transition, and the infrared duality between free fermions and $U(N)_1$ Chern-Simons matter theory with scalars.

Thank you